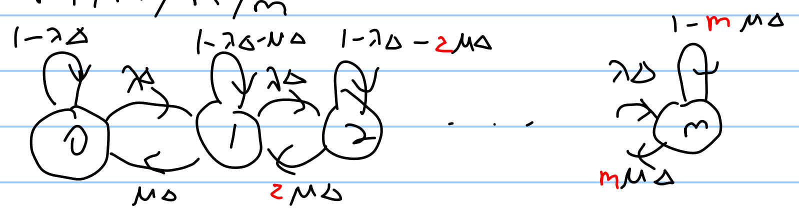
M/M/m/m



At equilibrium,

$$\pi_1(\mu\Delta) = \pi_0(\lambda\Delta)$$

$$\pi_1 = \rho\,\pi_0$$

$$\pi_2(2\mu\Delta) = \pi_1(\lambda\Delta)$$

$$\pi_2 = \frac{\rho}{2}\pi_1 = \frac{\rho^2}{2}\pi_0$$

...

$$\pi_m(m\mu\Delta) = \pi_{m-1}(\lambda\Delta)$$

$$\pi_m = \frac{\rho}{m}\pi_{m-1} = \frac{\rho^m}{m!}\pi_0$$

$$\pi_0 + \pi_1 + \pi_2 + \cdots \pi_m = 1$$

$$\Rightarrow \pi_0\left(1 + \rho + \frac{\rho^2}{2} \cdots \frac{\rho^m}{m!}\right) = 1$$

$$\pi_k = \frac{\rho^k}{k!}\pi_0 = \frac{\rho^k/k!}{\sum_{n=0}^{m} \rho^n/n!} \leftarrow \text{Erlang B formula}$$

M/G/1

For a general service distribution, Markov Chain cannot be used

Assume 1st + 2nd order statistics of the server are given.

$$\bar{X} = \frac{1}{\mu} \qquad \overline{X^2} = E[X^2]$$

We can obtain the average wait time W by Pollaczek-Khinchin (P-K) formula:

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)}$$

Proof: $W_i = \sum_{j=i-N_i}^{i-1} X_j + R_i$

W_i: wait time for the ith arrival

$\sum_{j=i-N_i}^{i-1} X_j$: service time for N_i arrival that are in queue

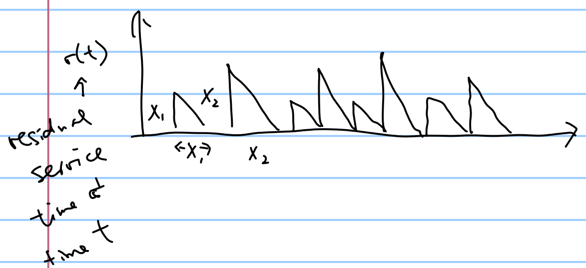
R_i: Residual service time: the time remaining to serve the one in service

Take expectation on both side

$$W = N_Q \frac{1}{\mu} + R$$

W: average wait time

R: average residual time



$$R = \frac{1}{t}\int_0^t r(\tau)\,d\tau$$

$$= \frac{M(t)}{t}\,\frac{1}{M(t)}\sum_{n=1}^{M(t)} \frac{X_i^2}{2}$$

$M(t)$: # of arrivals that have been serviced

$$= \frac{\lambda \overline{X^2}}{2}$$

$$\therefore \quad W = \frac{N_Q}{\mu} + \frac{\lambda \overline{X^2}}{2}$$

but $N_Q = \lambda W$ by Little's Thm

$$\therefore \quad W = \rho W + \lambda \overline{X^2}/2$$

$$\Rightarrow W = \frac{\lambda \overline{X^2}}{2(1-\rho)} \quad \blacksquare$$

Eg. M/D/1

$$\overline{X^2} = \frac{1}{\mu^2} \qquad \therefore W = \frac{\lambda}{2\mu^2(1-\rho)} \qquad + \qquad T = W + \frac{1}{\mu} = \frac{2-\rho}{2(\mu-\lambda)}$$